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UNSTEADY COMBINED CONDUCTION–RADIATION ENERGY TRANSFER USING A RIGOROUS DIFFERENTIAL METHOD

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Abstract—The transient energy transfer by simultaneous conduction and radiation in a thermal radiation absorbing, emitting and scattering medium is investigated analytically. The medium is confined between two gray, diffuse, isothermal planes kept at different but uniform temperatures. The problem is formulated rigorously in terms of a nonlinear fourth order differential equation. The complexity of the analysis for the conventional exact integral formulation is tremendously reduced by introducing this rigorous differential formulation. The differential formulation is also found to lend itself more readily to the different limiting and special cases. The numerical results are obtained by using an implicit finite difference method.

The temperature distributions are evaluated and compared with the steady-state results.

NOMENCLATURE

B, integrated Planck function, $B = \frac{\sigma}{\pi} T^4$ [Btu/hft^2];

c, speed of light [ft/h];

C, specific heat [Btu/lb$_m$°R];

E_n, exponential integral of order n, $E_n(\tau) = \int_0^1 e^{-\tau/\mu} \mu^{n-2} d\mu$;

I_1, I_2, outgoing intensities at wall 1 and wall 2 respectively, $I(0)$, $I(\tau_L)$;

I, intensity, radiant energy flowing in the direction (l, m, n) per unit of time, of solid angle, and of surface area normal to (l, m, n) [Btu/hft^2/μ];

I^+, dimensionless intensity, $I^+ = I/\sigma T_1^4$;

J, mean intensity, $J = \frac{1}{4\pi} \int_{4\pi} I \, d\omega$ [Btu/hft^2];

k, thermal conductivity [Btu/hft°R];

L, thickness of the plane layer [ft];

l, m, n, direction cosines;

N, conduction–radiation interaction parameter, $N = k\beta/4\sigma T^3$;

p, element of the radiative pressure tensor, $p = \frac{1}{c} \int_{4\pi} I\mu^2 d\omega$ [Btu/ft^3];

q, total energy flux [Btu/hft^2];

q_c, conduction heat flux [Btu/hft^2];

q_r, radiative energy flux, $q_r = 2\pi \int_{-1}^{1} I d\mu$ [Btu/hft^2];

q^+, dimensionless total energy flux, $q^+ = \frac{q}{\sigma T_1^4}$;

S, source function $S = \lambda B + (1 - \lambda) J$ [Btu/hft$^2\mu$];

S^+, dimensionless source function, $S^+ = \pi S/\sigma T_1^4$;

t, time [h];
t', Fourier modulus, $t' = \frac{\alpha t}{L^2}$;
T, temperature [°R];
x, coordinate perpendicular to the boundary [ft].

Greek symbols
α, thermal diffusivity [ft^2/h];
β, extinction coefficient (absorption coefficient + scattering coefficient) [ft^{-1}];
ε, emissivity of wall surface;
η, dimensionless parameter defined in equation (5);
θ, dimensionless temperature, (T/T_1);
λ, ratio of absorption to extinction coefficient, $\lambda = 1 - \tilde{\omega}_0$;
μ, directional cosine between x and I;
ξ, dimensionless coordinate, (x/L);
ρ, density [lb$_m$/ft^3];
σ, Stefan–Boltzmann constant ($0{\cdot}1714 \times 10^{-8}$) [Btu/hft^2°R^4];
τ, optical depth, $\tau = \beta x$;
τ_L, optical thickness of plane layer, $\tau_L = \beta L$;
ω, solid angle;
$\tilde{\omega}_0$, albedo for single scattering (scattering coefficient/extinction coefficient).

Subscripts
1,2, refer to wall 1 and wall 2 respectively;
c,r, refer to conduction and radiation respectively.

Superscript
+, denotes dimensionless quantity.

INTRODUCTION

LICK [1], studying the transient energy transfer problem of simultaneous conduction and radiation in a semi-infinite gray medium, presented asymptotic approximations, for short and long periods of time, for two cases corresponding to the presence and absence of external radiative flux. Nemchinov [2] investigated a similar problem where the two-flux approximation for radiative transfer was employed. In 1967 unsteady energy transfer in a plane layer of radiating (nonconducting) stagnant gray gas, where physical properties varied with temperature, was analyzed by Viskanta and Bathla [3]. More recently Heinisch and Viskanta [4], using an approximate analysis, have investigated the problem of transient combined conduction and radiation heat transfer in a semi-infinite gray optically thick medium with variable thermophysical and radiative properties.

PROBLEM

The physical model and the coordinate system for the present problem are shown schematically in Fig. 1. Two opaque gray diffuse parallel walls each with a different uniform temperature are indicated. Between them is a gray stagnant medium that conducts heat as well as absorbs, emits and scatters radiant energy. The index of refraction of the medium is considered to be unity, and all the relevant properties are assumed to be independent of temperature and wavelength.

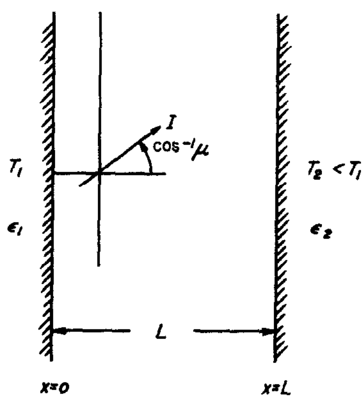

FIG. 1. Schematic diagram of physical system.

ANALYSIS

The transient one-dimensional energy equation for a stagnant radiating medium, in the absence of heat sources, may be written as

$$-\rho C\frac{\partial T}{\partial t} = -k\frac{\partial^2 T}{\partial x^2} + \frac{\partial q_r}{\partial x}. \tag{1}$$

The transfer equation [5] may be integrated over 4π rad after multiplying through by 1 and μ. The results are, respectively,

$$\frac{\partial q_r}{\partial x} = 4\pi\beta\lambda(B - J) \tag{2}$$

and

$$q_r = -\frac{c}{\beta}\frac{\partial p}{\partial x}. \tag{3}$$

Combining equations (1) and (2), we obtain

$$J = B - \frac{1}{4\pi\beta\lambda}\left(k\frac{\partial^2 T}{\partial x^2} - \rho C\frac{\partial T}{\partial t}\right). \tag{4}$$

We now introduce a dimensionless parameter relating the photon pressure p and the mean intensity J as follows:

$$\eta = \frac{3}{4\pi}\frac{cp}{J}. \tag{5}$$

Equation (4) indicates that η is equal to unity for half-range isotropic intensity distribution in both directions. It can also be shown that $\eta \approx 1$ for both the optically thick and thin limits [6]. Equations (3) and (4) yield

$$q_r = -\frac{4\pi}{3\beta}\frac{\partial}{\partial x}(\eta J). \tag{6}$$

Combining equations (4) and (6), we obtain

$$q_r = -\frac{4\pi}{3\beta}\frac{\partial}{\partial x}\left[\eta B - \frac{\eta}{4\pi\beta\lambda}\left(k\frac{\partial^2 T}{\partial x^2} - \rho C\frac{\partial T}{\partial t}\right)\right]. \tag{7}$$

Differentiating equation (7) with respect to x, energy equation (1) becomes

$$\rho C\frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial x^2} + \frac{4\pi}{3\beta}\frac{\partial^2}{\partial x^2}(\eta B) - \frac{1}{2\beta^2\lambda}\frac{\partial^2}{\partial x^2}\left(k\eta\frac{\partial^2 T}{\partial x^2} - \rho C\eta\frac{\partial T}{\partial t}\right). \tag{8}$$

Introducing the following dimensionless parameters:

$$\xi = \frac{x}{L}, \quad \theta = \frac{T}{T_1}, \quad N = \frac{k\beta}{4\sigma T_1^3}, \quad t' = \frac{\alpha t}{L^2}. \tag{9}$$

Equations (8) and (7) in dimensionless form become, respectively,

$$\frac{\partial\theta}{\partial t'} = \frac{\partial^2\theta}{\partial\xi^2} + \frac{1}{3N}\frac{\partial^2}{\partial\xi^2}(\eta\theta^4) - \frac{1}{3\lambda\tau_L^2}\frac{\partial^2}{\partial\xi^2}\left(\eta\frac{\partial^2\theta}{\partial\xi^2}\right) + \frac{1}{3\lambda\tau_L^2}\frac{\partial^2}{\partial\xi^2}\left(\eta\frac{\partial\theta}{\partial t'}\right), \tag{10}$$

$$\frac{3\tau_L}{4}q_r^+ = -\frac{\partial}{\partial\xi}(\eta\theta^4) + \frac{N}{\lambda\tau_L^2}\frac{\partial}{\partial\xi}\left(\eta\frac{\partial^2\theta}{\partial\xi^2}\right) - \frac{N}{\lambda\tau_L^2}\frac{\partial}{\partial\xi}\left(\eta\frac{\partial\theta}{\partial t'}\right). \tag{11}$$

Equation (10) is a non-linear fourth-order partial differential equation. Notice, however, that η contained in equation (10) requires integration. This equation must be solved with four appropriate boundary conditions. Two boundary conditions are

$$\theta(0, t') = 1$$
$$\theta(1, t') = \theta_2 = \text{constant}. \tag{12}$$

Two more boundary conditions can be obtained by using the radiosities at both walls. The radiation balance results in [6]:

$$q_{r_1}^+ = \varepsilon_1 - 2\varepsilon_1\left(\theta_2^4 - \frac{1-\varepsilon_2}{\varepsilon_2}q_{r_2}^+\right)E_3(\tau_L) - 2\varepsilon_1\int_0^{\tau_L} S^+(\tau')\,E_2(\tau')\,d\tau', \tag{13}$$

$$q_{r_2}^+ = -\varepsilon_2\theta_2^4 + 2\varepsilon_2\left(1 - \frac{1-\varepsilon_1}{\varepsilon_1}q_{r_1}^+\right)E_3(\tau_L) + 2\varepsilon_2\int_0^{\tau_L} S^+(\tau')\,E_3(\tau_L - \tau')\,\mathrm{d}\tau'. \quad (14)$$

Initially we set the temperature at T_1, i.e.

$$\theta(\xi, 0) = 1. \quad (15)$$

Equations (12)–(15) constitute the complete set of initial and boundary conditions for fourth-order differential equation (10) for which we now list some limiting cases.

(1) $N \gg 1$ and $\tau_L \gg 1$

Equation (10) reduces to the one-dimensional heat conduction equation.

(2) Optically thick limit ($\tau_L \gg 1$)

$$\frac{\partial\theta}{\partial t'} = \frac{\partial^2}{\partial\xi^2}\left(\theta + \frac{\eta\theta^4}{3N}\right). \quad (16)$$

For $\eta = 1$, equation (16) indicates that the net total heat flux is the sum of heat transfer by pure conduction and by pure radiation (as given by the Rosseland approximation).

(3) Conduction predominant case ($N \gg 1$)

Equation (10) becomes

$$\frac{\partial\theta}{\partial t'} = \frac{\partial^2\theta}{\partial\xi^2} - \frac{1}{3\lambda\tau_L^2}\frac{\partial^2}{\partial\xi^2}\left(\eta\frac{\partial^2\theta}{\partial\xi^2}\right) + \frac{1}{3\lambda\tau_L^2}\frac{\partial^2}{\partial\xi^2}\left(\eta\frac{\partial\theta}{\partial t'}\right). \quad (17)$$

(4) The case of weakly interacting system ($N/\lambda\tau_L^2 \gg 1$) and ($N \leqslant 1$). This implies $1 \geqslant N \gg \lambda\tau_L^2$.

Equation (10) reduces to

$$\frac{\partial^2}{\partial\xi^2}\left(\eta\frac{\partial\theta}{\partial t'}\right) = \frac{\partial^2}{\partial\xi^2}\left(\eta\frac{\partial^2\theta}{\partial\xi^2}\right) \quad (18)$$

which yields the solution directly when integrated twice.

(5) Optically thin limit and radiation predominant system ($\tau_L \ll 1$ and $N/\lambda\tau_L^2 \approx 1$). This implies $N \ll \lambda$.

$$\frac{\partial^2}{\partial\xi^2}\left\{\eta\left[\theta^4 - \frac{N}{3\lambda\tau_L^2}\frac{\partial^2\theta}{\partial\xi^2} + \frac{N}{3\lambda\tau_L^2}\frac{\partial\theta}{\partial t'}\right]\right\} = 0. \quad (19)$$

RESULTS AND DISCUSSION

Where both N and τ_L^2 are large compared with unity, the temperature distribution approaches that for pure conduction as in the $N = \tau = 1$ results shown in Fig. 2. For a large value of τ_L, 10, and a small value of N, 0·01, the results are depicted in Fig. 3; this problem can be solved by using the Rosseland approximation, see equation (16). Keeping the same N value, 0·01, and reducing the τ_L to 1 gives the results that are shown in Fig. 4 which look quite unlike the pure conduction results in terms of shape and dimensionless time required to approach steady state. Another case not greatly different from pure conduction is that where $\tau_L = 1{\cdot}0$ and $N = 0{\cdot}1$ (Fig. 5).

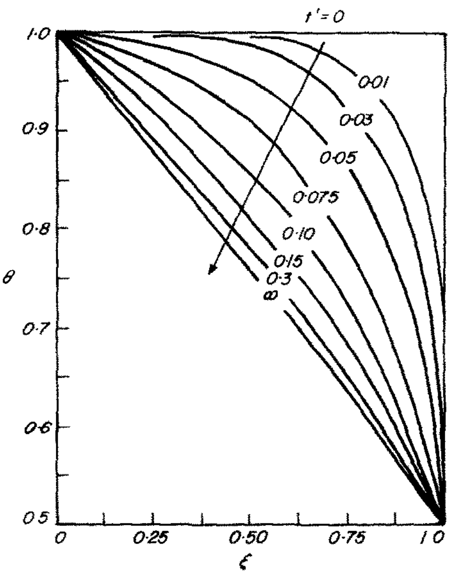

FIG. 2. Variation of dimensionless temperature with dimensionless thickness for $\theta_2 = 0{\cdot}5$, $\tau_L = 1{\cdot}0$ and $N = 1{\cdot}0$.

Most of these transient results differ from those of simple conduction in that the steady state is approached more rapidly. Also, instead of the diffusion type of heat flow where the temperature variation occurs only near the $\xi = 1$ boundary for small values of t', the temperature change penetrates deeply. This could be

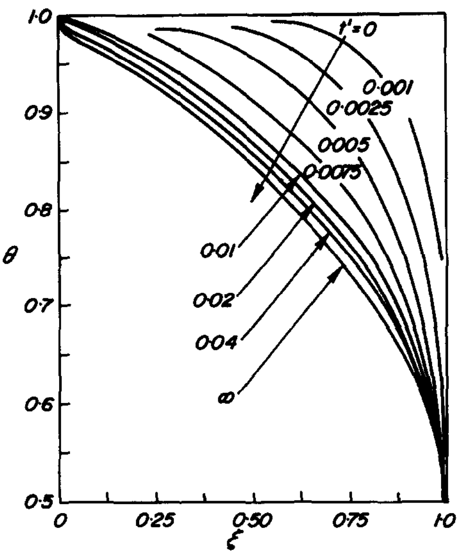

FIG. 3. Variation of dimensionless temperature with dimensionless thickness for $\theta_2 = 0{\cdot}5$, $\tau_L = 10$, $N = 0{\cdot}01$.

investigated further by examining the heat flux at $\xi = 0$ [6]. The value of η tends to be near unity for a number of cases; thus the calculation procedure can sometimes be simplified by letting $\eta = 1$ [6].

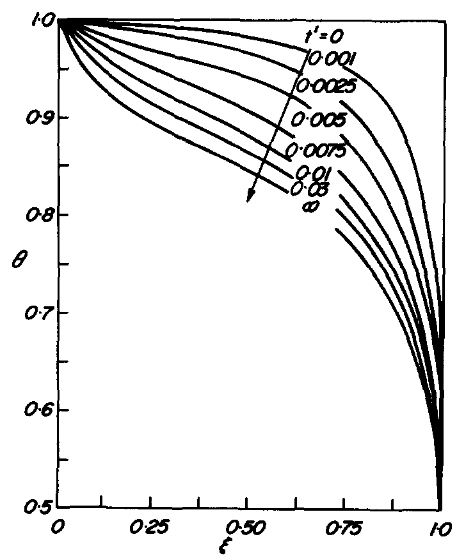

FIG. 4. Variation of dimensionless temperature with dimensionless thickness for $\theta_2 = 0{\cdot}5$, $\tau_L = 1{\cdot}0$ and $N = 0{\cdot}01$.

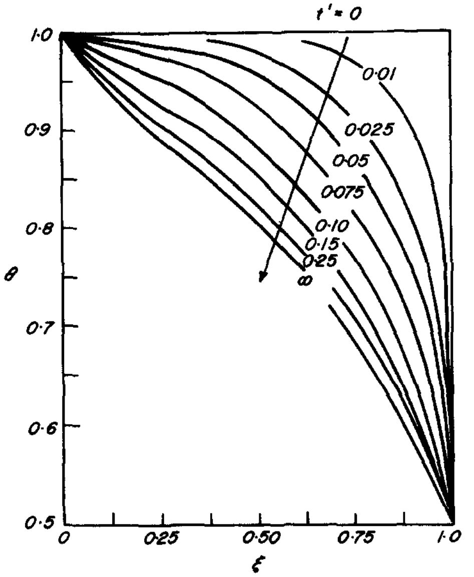

FIG. 5. Variation of the dimensionless total heat transfer with dimensionless time for $\theta_2 = 0{\cdot}5$, $\tau_L = 1{\cdot}0$, and $N = 0{\cdot}1$.

CONCLUSIONS

By analytically investigating the transfer of transient energy by simultaneous conduction and radiation in an absorbing, emitting, and scattering medium, and by formulating the problem in terms of a non-linear fourth-order differential equation into which a coefficient involving integration has been introduced, we have come to these conclusions:

(a) The complexity of the analysis for the conventional integral formulation [3] is tremendously reduced by the introduction of this rigorous differential method,

(b) The resulting differential formulation is found to lend itself more readily to the various limiting and special cases,

(c) The unique quality of this analysis is that it has the advantage of being particularly adaptable to digital solution and to extension to other more difficult geometries.

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TRANSPORT D'ÉNERGIE INSTATIONNAIRE AVEC COMBINAISON DE LA CONDUCTION ET DU RAYONNEMENT EN EMPLOYANT UNE MÉTHODE DIFFÉRENTIELLE RIGOUREUSE

Résumé—Le transport d'énergie transitoire par conduction et rayonnement simultanés est étudié analytiquement dans un milieu absorbant, émettant et diffusant le rayonnement thermique. Le milieu est confiné entre deux plans gris, diffus et isothermes gardés à des températures uniformes mais différentes. Le problème est formulé rigoureusement sous la forme d'une équation différentielle non-linéaire du quatrième ordre. La complexité de l'analyse pour la formulation intégrale exacte classique est réduite considérablement en introduisant cette formulation différentielle rigoureuse. On trouve aussi que la formulation différentielle se prête elle-même plus facilement aux différente cas spéciaux et limites. Les résultats numériques sont obtenus en employant une méthode implicite de différences finies. Les distributions de température sont évaluées et comparées avec les résultats en régime permanent.

BERECHNUNG DES INSTATIONÄREN WÄRMETRANSPORTS DURCH LEITUNG UND STRAHLUNG IM GLEICHEN FELD NACH EINER STRENGEN DIFFERENTIALMETHODE

Zusammenfassung—In einem, Temperaturstrahlung absorbierenden, emittierenden und streuenden Medium wird der gleichzeitig durch Leitung und Strahlung bewirkte Energietransport analytisch untersucht.

Das Medium wird durch zwei, grau und diffus reflektoerende, isotherme Flächen begrenzt, welche auf verschiedenen Temperaturen gehalten werden. Das Problem wird streng durch eine nichtlineare Differentialgleichung vierter Ordnung beschrieben. Die grossen mathematischen Schwierigkeiten bei Lösung dieser Aufgabe in der üblichen Weise, als Integralgleichungsproblem, werden durch die Einführung der Differentialform wesentlich verringert. Die Differentialform erweist sich auch als geeignteter zur Untersuchung von Grenz- und Spezialfällen.

Die numerischen Resultate werden mit Hilfe eines impliziten Differenzenverfahrens erhalten. Die errechneten Temperaturverteilungen werden mit den Ergebnissen des stationären Falles verglichen.

ПЕРЕДАЧА НЕУСТАНОВИВШЕЙСЯ ЭНЕРГИИ ОДНОВРЕМЕННО КОНДУКЦИЕЙ И ИЗЛУЧЕНИЕМ, ОПРЕДЕЛЕННОЙ СТРОГО ДИФФЕРЕНЦИАЛЬНЫМ МЕТОДОМ

Аннотация—Аналитическое исследование передачи неустановившейся энергии одновременно кондукцией и излучением в среде поглащающей, излучающей и рассеивающей тепловое излучение. Среда заключена между двумя нейтральными, диффундированными изометрическими плоскостями, температура которых различная, но постоянная. Проблема формулируется в строго нелинейном четвертом порядке дифференциального уравнения. Сложный анализ исчисления принятого точного интеграла, очень облегчается введением этой строго дифференциальной формулировки. Нашли, что дифференциальное исчеслецие также легче применять в различных крайних и специальных случаях. Численные результаты получили неявным финитовым разностным методом. Определили распределения температур и сравнили их с результатами установившегося режима.